POW 15
Growth of rat populations

Problem Statement

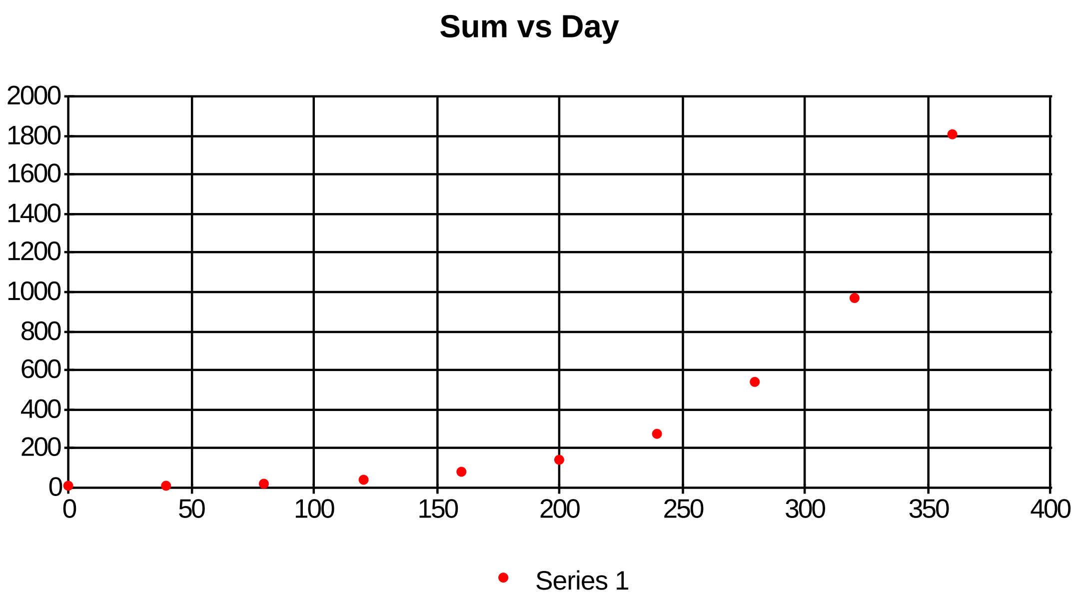
This problem is composed of the growth of a rat population over the course of one year from 2 rats. Four assumptions for this problem were made: Each new liter is composed of 6 rats; 3 males, 3 females. The original pair give birth to 6 rats on the first day and then ever 40 days after. There is a 120 day "gestation" period before a newborn rat can reproduce. After this gestation period the rats will give birth every 40 days. Also over the course of a year no rats die. The key question is how many rats are there after one year.

Process

In order to solve this problem I first focused on getting an answer for the number of rats. Originally I had planned to use a large sheet of poster board paper to draw out the family tree. But quickly realized that this would be quite impractical. I then settled on a similar but more efficient method. In this method I focused on the 10 liters that the original parents produced. The 10 liters comes from the fact that there is 365.25 days in a year and the parent liter produce every 40 days from day zero. Meaning that they will have a liter on day 0, 40, 80, 120, 160, 200, 240, 280, 320, and 360. I started on the day 360 liter and worked backwards. After finishing this method I compiled the data in a spreadsheet. From which I was able to sum each interval of rat births. The data collected in family tree method is shown in the following data table:

Couple:	**0**	**40**	**80**	**120**	**160**	**200**	**240**	**280**	**320**	**360**	**Total**	**Sum after birth**
Original	2										2	2
D0	6										6	8
D40		6									6	14
D80			6								6	20
D120	18			6							24	44
D160	18	18			6						42	86
D200	18	18	18			6					60	146
D240	72	18	18	18			6				132	278
D280	126	72	18	18	18			6			258	536
D320	180	126	72	18	18	18			6		438	974
D360	396	180	126	72	18	18	18			6	834	1808
Sum	*836*	*438*	*258*	*132*	*54*	*36*	*18*	6	6	6	1808	

I then found the pattern that emerged in the upper data table interesting. So I decided to look into it further. At first I looked at the graph of the sum after birth to see what kind of relationship it was.



As clearly shown the growth is exponential. I then tried to get a model for this data but could not get anything that fit it perfectly, which is what I was aiming for. So I moved on to looking for relationships in the data. I looked at the ratios and differences between the days and the sums, and rats born. But was unable to come up with anything. I then tried to look at other relationships between sterile rats, reproducible rats, and the sum before and after birth.

Sum after birth	# rats reproducable	sum before birth	# sterile
2			
8	2	2	0
14	2	8	6
20	2	14	12
44	8	20	12
86	14	44	30
146	20	86	66
278	44	146	102
536	86	278	192
974	146	536	390
1808	278	974	696

Solution

My solution for this problem was 1808 rats at the end of 1 year. I believe that this solution is correct because of the pattern noticed when looking at the compiled data table. Also because after creating the table shown above (last page) it was logical that that was how it worked out. Through the course of working on the problem I looked at various ways of solving and trying to find patterns and became familiar enough with the growth of the rats that my answer appears to be completely logical with no questions or discrepancies left open.

Generalizations

In order to generalize this problem I first looked into jumping straight to an explicit formula but quickly found that doing so would be highly improbable. So I then looked into a recursive function that would model the population. Looking at the growth logically I noticed that the first 3 dates 6 rats were due. Then on the 4th date (120) the number of rats born were going to be 6+ the number born 120 days earlier, or 3 dates back. Then if you continue this a recursive function emerges; such that:

$U_0 = 6$
$U_1 = 6$
$U_2 = 6$
$U_n = \left(3(U_{n-3}) + 3(U_{n-4}) + 3(U_{n-5}) + ... + 3(U_{n-n})\right) + 6$

What the above recursive function gives you is the number of rats born on the nth day, in which:

Day	nth
0	0
40	1
80	2
120	3
160	4
200	5
240	6
280	7
320	8
360	9
n*40	n

Because this function only gave the number of rats born on a particular day another function had to be created that would sum the day given and previous dates. The work that leads to this function is as follows:

The first function works as such:

$$U_0 = 6$$
$$U_1 = 6$$
$$U_2 = 6$$
$$U_3 = 3(U_{n-3}) + 6$$
$$U_4 = 3(U_{n-3}) + 3(U_{n-4}) + 6$$
$$U_5 = 3(U_{n-3}) + 3(U_{n-4}) + 3(U_{n-5}) + 6$$
$$U_6 = 3(U_{n-3}) + 3(U_{n-4}) + 3(U_{n-5}) + 3(U_{n-6}) + 6$$
$$U_7 = 3(U_{n-3}) + 3(U_{n-4}) + 3(U_{n-5}) + 3(U_{n-6}) + 3(U_{n-7}) + 6$$
$$U_8 = 3(U_{n-3}) + 3(U_{n-4}) + 3(U_{n-5}) + 3(U_{n-6}) + 3(U_{n-7}) + 3(U_{n-8}) + 6$$

If you sum these:

$$U_0 = 6 + 2$$
$$U_1 = \left(6 + (U_{n-1})\right)$$
$$U_2 = \left(6 + (U_{n-1}) + (U_{n-2})\right)$$
$$U_3 = \left(3(U_{n-3}) + 6 + ((U_{n-1}) + (U_{n-2}) + (U_{n-3}))\right)$$
$$U_4 = \left(3(U_{n-3}) + 3(U_{n-4}) + 6 + ((U_{n-1}) + (U_{n-2}) + (U_{n-3}) + (U_{n-4}))\right)$$
$$U_5 = \left(3(U_{n-3}) + 3(U_{n-4}) + 3(U_{n-5}) + 6 + ((U_{n-1}) + (U_{n-2}) + (U_{n-3}) + (U_{n-4}) + (U_{n-5})\right)$$

From this you can make the general recursive formula

$$U_0 = 6$$
$$U_1 = 12$$
$$U_2 = 18$$
$$U_n = \left(\left(3(U_{n-3}) + 3(U_{n-4}) + 3(U_{n-5}) + \ldots + 3(U_{n-n})\right) + 6 + ((U_{n-1}) + (U_{n-2}) + (U_{n-3}) + (U_{n-4}) + (U_{n-5}) + (U_{n-n})\right) + 2$$

This can be simplified to:

$$U_0 = 6$$
$$U_1 = 12$$
$$U_2 = 18$$
$$U_n = (U_{n-1}) + (U_{n-2}) + \left(4(U_{n-3}) + 4(U_{n-4}) + 4(U_{n-5}) + \ldots + 4(U_{n-n})\right) + 8$$

Which can be used as a recursive function to determine the number of rats at any time in 40 day increments for the assumptions given.

This can be generalized further:

$$U_0 = A$$
$$U_1 = 2 * A$$
$$U_2 = 3 * A$$
$$U_n = \left(\left(\frac{A}{2}(U_{n-\frac{G}{I}}) + \frac{A}{2}(U_{n-(\frac{G}{I}+1)}) + \frac{A}{2}(U_{n-(\frac{G}{I}+2)}) + \ldots + \frac{A}{2}(U_{n-n})\right) + A + ((U_{n-1}) + (U_{n-2}) + (U_{n-3}) + (U_{n-4}) + (U_{n-5}) + (U_{n-n})\right) + 2$$

Where:

A= the number of rats born in a liter

G= the "gestation" period that a newborn rat must wait before giving birth

I= the interval between births

Self-Assessment

On this POW I worked very diligently starting on this last Sunday and working on it a little bit each day. I started with getting an answer and then worked on generalizing that answer so more could be attained. I believed I learned better methods of creating a recursive formula for a more complicated situation then I had worked with in the past. I also gained more confidence in my ability to solve problems with this. I would have liked to have come up with an explicit function but given the time constraints I was unable to. The only help that I received with this POW was checking my final total number with other students to ensure that I had not made any errors that would result in making the generalization invalid.